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On a Class of Interpolatory Splines

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Several properties of a class of interpolatory splines are studied. This class is a generalization of the quadratic splines of M. J. Marsden.

1. INTRODUCTION

Recently Marsden [2] studied the properties of quadratic splines which interpolate, given functions at the midpoints between knots. He proved that the linear interpolation projection mappings are uniformly bounded *independent* of any assumption on the partition, unlike several known cubic spline interpolation mappings.

On the other hand, de Boor [1] has shown that the same property holds for cubic splines which interpolate at the average of three consecutive knots.

Earlier, Meir and Sharma [4] studied properties of deficient cubic splines, interpolating at two points in each subinterval formed by the knots, with the points of interpolation following the same pattern in each subinterval. Their result implies that this type of interpolation mapping is also uniformly bounded independent of any assumption on the partition.

Our aim is to define and study the properties of a class of splines of arbitrary degree, which includes as special cases the classes introduced by Marsden [2], and Meir and Sharma [4]. This study is performed by using Meir and Sharma's method [3].

The paper contains the proof of existence and uniqueness properties of an interpolation problem involving a finite number of knots and between two consecutive knots a number of interpolatory conditions, following the same pattern in each subinterval.

2. NOTATIONS, DEFINITIONS AND PRELIMINARIES

Let N be a positive integer greater or equal to 2. For $-\infty < a < b < \infty$, let

$$\Delta := x_0 < x_1 < \cdots < x_N = b \tag{2.1}$$

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denote a partition of $[a, b]$ with knots x_i. The set of all such partitions is denoted by $P(a, b)$. Another notation used is $h_i = x_i - x_{i-1}$, $i = 1,..., N$.

Let π_n be the set of all real algebraic polynomials of degree at most n. We define the polynomial spline class $S_{n,\Delta}$ as

$$S_{n,\Delta} = \{s(x) \mid s(x) \in C^1[a, b], s(x) \in \pi_n \text{ for } x \in (x_{i-1}, x_i), i = 1, 2,..., N\}. \tag{2.2}$$

We shall use also the notation $C^{-1}[a, b]$ for the space of all bounded function on $[a, b]$.

Let $0 < \theta_1 < \cdots < \theta_{n-1} < 1$ be a given subdivision of $[0, 1]$ and define $\omega(\theta) = \prod_{j=1}^{n-1} \theta - \theta_j)$.

Denote

$$t_{ij} = x_{i-1} + \theta_j h_i, \qquad i = 1,..., N; \quad j = 1,..., n-1. \tag{2.3}$$

Now we can formulate the interpolation problem we want to study.

PROBLEM. *Let $f_{ij} (i = 1,..., N; j = 1,..., n-1)$ be an arbitrary set of real numbers. Find a function $s(x)$ in $S_{n,\Delta}$ satisfying*

$$s(t_{ij}) = f_{ij}, \qquad i = 1,..., N; \quad j = 1,..., n-1. \tag{2.4}$$

The special case $n = 2$ with $\theta_1 = \frac{1}{2}$ was treated by Marsden [2] and the case $n = 2$, $0 < \theta_1 < 1$, $\theta_1 \neq \frac{1}{2}$ by Sharma and Tzimbalario [5]. Meir and Sharma solved the problem for $n = 3$.

3. The Existence and Uniqueness

We shall use the method introduced by Meir and Sharma [3, 4]. We can state

THEOREM. *Let $f \in C^{-1}[a, b]$ and write $f_{ij} = f(t_{ij})$ for $i = 1, 2,..., N$; $j = 1, 2,..., n-1$. There is a unique element $s(x)$ of $S_{n,\Delta}$ satisfying* (2.4), *$s(a) = f(a)$ and $s(b) = f(b)$.*

The linear projector $P_\Delta : C^{-1}[a, b] \to S_{n,\Delta}$ which carries f to s is bounded:

$$\| P_\Delta \| \leqslant C(\omega), \tag{3.1}$$

where $C(\omega)$ is a positive constant depending only on ω. In the case

$$\theta_j = 1 - \theta_{n-j}, \qquad j = 1,..., n-1, \tag{3.2}$$

we have

$$C(\omega) = \frac{[\omega(1)]^2}{2\omega'(1)} \sum_{j=1}^{n} \frac{\| \omega \|_\infty}{\theta_j^2(1-\theta_j)^2 \mid \omega'(\theta_j)\mid} + \frac{1}{4} \sum_{j=1}^{n-1} \frac{\| \omega' \|_\infty}{\theta_j(1-\theta_j) \mid \omega(\theta_j)\mid}. \quad (3.3)$$

Note that P_Δ is bounded independently of Δ.

Proof. Let $s_i = s(x_i)$, $i = 0, 1, \ldots, N$ and set

$$\Omega_i(x) = (x - x_{i-1})(x - x_i)\, \omega\left(\frac{x - x_{i-1}}{h_i}\right). \quad (3.4)$$

For $x_{i-1} \leqslant x \leqslant x_i$, we have the following representation of the element of $S_{n,\Delta}$, which satisfies the interpolatory conditions (2.4).

$$s(x) = s_{i-1} \frac{\Omega_i(x)}{(x - x_{i-1})\, \Omega'_i(x_{i-1})} + s_i \frac{\Omega_i(x)}{(x - x_i)\, \Omega'_i(x_i)}$$

$$+ \sum_{j=1}^{n-1} f_{ij} \frac{\Omega_i(x)}{(x - t_{ij})\, \Omega'_i(t_{ij})} \equiv s_{i-1}\lambda_{0,i}(x)$$

$$+ s_i\lambda_{n,i}(x) + \sum_{j=1}^{n-1} f_{ij}{}^{n}{}_{ji}(x). \quad (3.5)$$

The numbers s_i, $i = 0, 1, \ldots, N$, will be determined by the continuity of the first derivative at the knots x_i, $i = 1, 2, \ldots, N-1$, i.e., $s'(x_i-) = s'(x_i \times)$. Hence, we get

$$s_{i-1}\lambda'_{0i}(x_i) + s_i[\lambda'_{n,i}(x_i) - \lambda'_{0,i+1}(x_i)] - s_{i+1}\lambda'_{n,i+1}(x_i)$$

$$= \sum_{j=1}^{n-1} f_{i+1,j}\lambda_{j,i+1}(x_i) - \sum_{j=1}^{n-1} f_{i,j}\lambda_{j,i}(x_i). \quad (3.6)$$

By some elementary manipulations, system (3.6) reduces to

$$s_0 = f(a), \qquad S_N = f(b),$$

$$\frac{s^i{}_{-1}}{h_i} + \left\{\left[1 + \frac{\omega'(1)}{\omega(1)}\right]\frac{1}{h_i} + \left[1 - \frac{\omega'(0)}{\omega(0)}\right]\frac{1}{h_{i+1}}\right\} s_i + \frac{s_{i+1}}{h_{i+1}} \quad (3.7)$$

$$= \omega(0) \sum_{j=1}^{n-1} \frac{f_{i+1,j}}{h_{i+1}\theta_j^2(1-\theta_j)\, \omega'(\theta_j)} + \omega(1) \sum_{j=1}^{n-1} \frac{f_{i,j}}{h_i\theta_j(1-\theta_j)^2\, \omega'(\theta_j)}.$$

It is obvious that $\omega'(1)/\omega(1) > 0$ and $-\omega'(0)/\omega(0) > 0$, so the matrix corresponding to (3.7) is strictly diagonally dominant, and this proves that (3.7) has a unique solution.

Let $\mid s_k \mid = \text{mas}\{\mid s_i \mid, i = 0, 1, \ldots, N\}$. If $j = 0$, or $j = N$, clearly

$\max\{|s_i|,\ i = 0,..., N\} \leqslant \|f\|_\infty$. On the other hand, if $1 \leqslant j \leqslant N-1$, we easily have from (2.5) and (3.7), with $i = k$,

$$\{\max |s_i|, i = 0, 1,..., N\} \leqslant C(\omega) \|f\|_\infty, \tag{3.8}$$

In the special case (3.2) we have for some positive constant $C(\omega)$ which depends only on ω.

$$\left[1 + \frac{\omega'(1)}{\omega(1)}\right] |s_k| \leqslant |s_k| + \|f\|_\infty \frac{\omega(1)}{2} \left[\sum_{j=1}^{n-1} \frac{1}{\theta_j^2(1-\theta_j)^2 |\omega'(\theta_j)|}\right], \tag{3.9}$$

and hence

$$C(\omega) = \frac{[\omega(1)]^2}{2\omega'(1)} \sum_{j=1}^{n-1} \frac{1}{\theta_j^2(1-\theta_j)^2(\omega'(\theta_j))}. \tag{3.10}$$

In order to complete the result we consider an arbitrary subinterval (x_{i-1}, x_i). There, the interpolation function can be written as

$$\begin{aligned} s(x) &= \omega\left(\frac{x - x_{i-1}}{h_i}\right)\left\{-\frac{(x - x_i)}{\omega(0)\, h_i} s_{i-1} + \frac{(x - x_{i-1})}{\omega(1)\, h_i} s_i\right\} \\ &\quad + (x - x_{i-1})(x_i - x) \sum_{j=1}^{n-1} f_{ij} \frac{\omega[(x - x_{i-1})/h_i]}{\theta_j(1-\theta_j)(x - x_{i-1} - \theta_j h_i)\, \omega'(\theta_j)}. \end{aligned}$$

The mean value theorem and some elementary calculations will complete the proof.

REFERENCES

1. C. DE BOOR, On bounding spline interpolation, *J. Approximation Theory* **14** (1975), 191–203.
2. M. J. MARSDEN, Quadratic spline interpolation, *Bull. Amer. Math. Soc.* **80** (1974), 903–906.
3. A. MEIR AND A. SHARMA, Degree of approximation of spline interpolation, *J. Math. Mech.* **15** (1966), 759–768.
4. A. MEIR AND A. SHARMA, Convergence of a class of interpolatory splines, *J. Approximation Theory* **1** (1968), 243–250.
5. A. SHARMA AND J. TZIMBALARIO, Quadratic splines, *J. Approximation Theory* **19** (1977), 186–193.